Linear Homogeneous Recurrence Relation's Solution For complex roots of Characteristic Equation

Ex: Let's consider the recurrence relation be $a_n = 2a_{n-1} - 2a_{n-2}$*. Solve it.*

Sol^n*: Let's assume the solution be* $a_n = t^n . [t \neq 0]$

Now from the recurrence relation given we have, $t^n = 2t^{n-1} - 2t^{n-2}$.

- ⇨ $t^n - 2t^{n-1} + 2t^{n-2} = 0$
- ⇨ $t^{n-2}(t^2 - 2t + 2) = 0$
- ⇨ $t^2 - 2t + 2 = 0$
- ⇨ $t = 1 \pm i.$
- ⇨ $a_n = A(1+i)^n + B(1-i)^n$
- ⇨ $a_n = A\left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^n + \left(\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\right)^n$
- ⇨ $a_n = A\left(\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\right) + B\left(\left(\sqrt{2}\right)^n\left(\cos\frac{\pi n}{4} - i\sin\frac{n\pi}{4}\right)\right)$
- ⇨ $a_n = \alpha\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4} + \beta\left(\sqrt{2}\right)^n\sin\frac{n\pi}{4}.$

Here $\alpha = A + B, \beta = i(A - B).$

Note: $\beta \notin \mathbb{R}$.

But let's verify that $\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}$ *is a solution.*

Verification: $2(\left(\sqrt{2}\right)^{n-1}\cos\frac{(n-1)\pi}{4} - \left(\sqrt{2}\right)^{n-2}\cos\frac{(n-2)\pi}{4})$

$$= 2(\left(\sqrt{2}\right)^{n-1}\left(\cos\frac{n\pi}{4}\cos\frac{\pi}{4} + \sin\frac{n\pi}{4}\sin\frac{\pi}{4}\right) - \left(\sqrt{2}\right)^{n-2}(\cos\frac{n\pi}{4}\cos\frac{\pi}{2} + \sin\frac{n\pi}{4}\sin\frac{\pi}{2}))$$

= $\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}$ $[verified]$.

Similarly we can verify that $\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$ *is also a solution.*

Hence we can consider the linear combination of $\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}$ & $\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$ *as a solution to the given recurrence relation!*

Remark: *We can take* $\alpha, \beta \in \mathbb{R}$. *Hence we get the most general solution as* $a_n = \alpha\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4} + \beta\left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$.