V's Metric On Mn(n,R)

Let $A, B \in Mn(n, \mathbb{R})$. *Then we define a metric d on* $Mn(n, \mathbb{R})$ *as follow:*

$d(A, B) = \sup\{|a_{ij} - b_{ij}|\}\ \forall i, j; 1 \leq i, j \leq n$.

Show that GL(n,R) is dense subset of Mn(n,R).

Pf: Case[1], choose $A \in Mn(n, \mathbb{R})$ *and let* $detA \neq 0$.

Now we have $A = (C_1\ C_2\ C_3 \ldots \ldots\ C_n)$ *where* $C_i's$ *are column vectors which are linearly independent. Now we construct a sequence* $(A_k)_{k\geq 1}$ *defined as*

$A_k = \left(C_1\ C_2\ C_3 \ldots \frac{k}{k+1} C_r \ldots\ C_n\right)$. *Now easy to see that* $(A_k)_{k\geq 1} \subseteq GL(n, \mathbb{R})$.

Using V's metric we shall now show that $A_k \longrightarrow A$.

$$d(A_k, A) = \frac{a_{ir}}{k+1} \longrightarrow 0, as\ k \longrightarrow \infty, for\ some\ i, 1 \leq i \leq n.$$

Hence proved.

Case[2]: $detA = 0$.

In that scenario we have say $C_1, C_2, C_3 \dots \dots, C_s$ *many column vectors are linearly independent rest are linear combination of them. Now we form a sequence as follow:*

$$A_k = \left(C_1\ C_2\ C_3 \dots C_s\ C_{s+1} + \frac{1}{k} e_{s+1} \dots . C_n + \frac{1}{k} e_n\right),$$

where $e_j\ for\ s + 1 \leq j \leq n$ *are basis vectors of* $\mathbb{R}^n$ *such that they are linearly* *independent with the column vectors* $C_t\ for\ 1 \leq t \leq s$.

Now easy to check $(A_k)_{k\geq 1} \subseteq GL(n, \mathbb{R})$.

Using V's metric we shall now show that $A_k \longrightarrow A$.

$$d(A_k, A) = \frac{1}{k} \to 0, as\ k \to \infty.$$

Hence proved.

Now the arbitrariness of $A \in Mn(n, \mathbb{R})$ *proves the denseness of* $GL(n, \mathbb{R})$.